

An introduction to particle simulation of rare events

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Lecture 3

- 1 Introduction, motivations
- 2 Particle interpretations of Feynman-Kac models
- 3 Additive functionals
- 4 Normalizing constant estimation
- 5 Some references

- 1 Introduction, motivations
 - Some rare event problems
 - Stochastic models
 - Importance sampling techniques
 - The heuristic of particle methods
 - 3 types of occupation measures
- 2 Particle interpretations of Feynman-Kac models
- 3 Additive functionals
- 4 Normalizing constant estimation
- 5 Some references

Rare event analysis

- **Stochastic process** $X \oplus$ **Rare event** A :

$$\text{Proba}(X \in A) \quad \& \quad \text{Law}((X_0, \dots, X_t) \mid X \in A)$$

▷ **engineering/physics/biology/economics/finance** :

- *Finance* : ruin/default probabilities, financial crashes, eco. crisis,...
 - *engineering* : networks overload, breakdowns, engines failures,...
 - *Physics* : climate models, directed polymer conformations, particle in absorbing medium, ground states of Schroedinger models.
 - *Statistics* : tail probabilities, extreme random values.
 - *Combinatorics* : Complex enumeration problems.
- **Process strategies** \in **Rare event** \Rightarrow **Control and prediction**.

$$X_t = F_t(X_{t-1}, W_t) \rightarrow \text{Law}((W_0, \dots, W_t) \mid X \in A)$$

Only 2 Ingredients

- **1 Physical/biological/financial process** : queuing network, portfolio, volatility process, stock market evolutions, interacting/exchange economic models ...
- **1 Potential function (energy type, indicator, restrictions)**: critical level crossing, penalties functions, constraints subsets, performance levels, long range dependence...

Objectives

- Estimation of the probability of the rare event.
- Computing **the full distributions of the path of the process** evolving in the critical regime \rightsquigarrow **prediction** \oplus **control**.

Twisted Monte Carlo methods

$$\mathbb{P}(X \in A) = 10^{-10} \rightsquigarrow \text{Find } \mathbb{Q} \text{ s.t. } \mathbb{Q}(A) \simeq 1$$

Elementary Monte Carlo estimate X^i iid \mathbb{Q}

$$\mathbb{P}(A) := \int \frac{d\mathbb{P}}{d\mathbb{Q}}(x) \mathbf{1}_A(x) \mathbb{Q}(dx) \simeq \mathbb{P}^N(A) := \frac{1}{N} \sum_{1 \leq i \leq N} \frac{d\mathbb{P}}{d\mathbb{Q}}(X^i) \mathbf{1}_A(X^i)$$

$$\text{Variance} \simeq \int \frac{d\mathbb{P}}{d\mathbb{Q}}(x) \mathbf{1}_A(x) \mathbb{P}(dx) \quad (\text{ex. cross-entropy : } \mathbb{Q} \in \{\mathbb{Q}_a, a \in \mathcal{A}\})$$

Drawbacks

- Huge variance if \mathbb{Q} badly chosen \rightsquigarrow optimal choice $\mathbb{Q}(dx) \propto \mathbf{1}_A(x)\mathbb{P}(dx)$.
- Need to twist the original reference process X .
- Stochastic evolution $X = (X_0, \dots, X_n)$

$$\frac{d\mathbb{P}_n}{d\mathbb{Q}_n}(X) := \prod_{k=0}^n \frac{p_k(X_k|X_{k-1})}{q_k(X_k|X_{k-1})} \quad \text{degenerate product martingale w.r.t. } n$$

Flow of measure with increasing sampling complexity

- Rare event = **cascade/series of intermediate less-rare events**
(\uparrow energy levels, physical gateways, index level crossings).
- Conditional probability flow = **flow of optimal twisted measures**

$$n \rightarrow \eta_n = \text{Law}(\text{process} \mid \text{series of } n \text{ intermediate events})$$

- Rare event probabilities = Normalizing constants.

Particle methods

(Sampling a genealogical type default tree model \oplus % success or default)

- **Explorations/Local search propositions** of the solution space.
- **Branching-Selection** individuals $\in \uparrow$ critical regimes.

5 Examples of flow of target measures

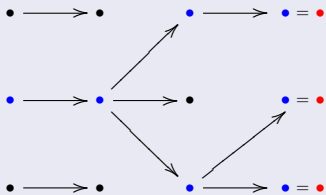
- 1 $\eta_n = \text{Loi}((X_0, \dots, X_n) \mid \forall 0 \leq p \leq n \quad X_p \in A_p)$
- 2 $\eta_n(dx) \propto e^{-\beta_n V(x)} \lambda(dx)$ with $\beta_n \uparrow$
- 3 $\eta_n(dx) \propto 1_{A_n}(x) \lambda(dx)$ with $A_n \downarrow$
- 4 $\eta_n = \text{Loi}_\pi^K((X_0, \dots, X_n) \mid X_n = x_n)$.
- 5 $\eta_n = \text{Loi}(X \text{ hits } B_n \mid X \text{ hits } B_n \text{ before } A)$

5 particle heuristics :

- 1 M_n -local moves \oplus individual selections $\in A_n$ i.e. $\sim G_n = 1_{A_n}$
- 2 MCMC local moves $\eta_n = \eta_n M_n \oplus$ individual selections $\propto G_n = e^{-(\beta_{n+1} - \beta_n)V}$
- 3 MCMC local moves $\eta_n = \eta_n M_n \oplus$ individual selections $\propto G_n = 1_{A_{n+1}}$
- 4 M -local moves \oplus Selection $G(x_1, x_2) = \frac{\pi(dx_2)K(x_2, dx_1)}{\pi(dx_1)M(x_1, dx_2)}$
- 5 M_n -local moves \oplus Selection-branching on upper/lower levels B_n .

Interaction/branch. process \hookrightarrow 3 types of occupation measures

($N = 3$)



- **Current population** $\hookrightarrow \frac{1}{N} \sum_{i=1}^N \delta_{\xi_n^i} \leftarrow i\text{-th individual at time } n$
- **Genealogical tree model** $\hookrightarrow \frac{1}{N} \sum_{i=1}^N \delta_{(\xi_{0,n}^i, \xi_{1,n}^i, \dots, \xi_{n,n}^i)} \leftarrow i\text{-th ancestral line}$
- **Complete genealogical tree model** $\hookrightarrow \frac{1}{N} \sum_{i=1}^N \delta_{(\xi_0^i, \xi_1^i, \dots, \xi_n^i)}$
- \oplus **Empirical mean potentials [success % ($G_n = 1_A$)]** $\hookrightarrow \frac{1}{N} \sum_{i=1}^N G_n(\xi_n^i)$

Equivalent Stochastic Algorithms :

- Genetic and evolutionary type algorithms.
- Spatial branching models.
- Sequential Monte Carlo methods.
- Population Monte Carlo models.
- Diffusion Monte Carlo (DMC), Quantum Monte Carlo (QMC), ...
- Some botanical names $\sim \neq$ application domain areas :
bootsrapping, selection, pruning-enrichment, reconfiguration, cloning, go with the winner, spawning, condensation, grouping, rejuvenations, harmony searches, biomimetics, splitting, ...



1950 \leq [(Meta)Heuristics] \leq 1996 \leq Feynman-Kac mean field particle model

- 1 Introduction, motivations
- 2 Particle interpretations of Feynman-Kac models
 - Quelques notations
 - Asymptotic Analysis
 - Nonlinear Markov chains
 - Mean field particle interpretations
 - Some cv. results
- 3 Additive functionals
- 4 Normalizing constant estimation
- 5 Some references

Some notation

E measurable state space, $\mathcal{P}(E)$ proba. on E , $\mathcal{B}(E)$ bounded meas. functions

- $(\mu, f) \in \mathcal{P}(E) \times \mathcal{B}(E) \longrightarrow \mu(f) = \int \mu(dx) f(x)$
- $M(x, dy)$ **integral operator over E**

$$M(f)(x) = \int M(x, dy) f(y)$$

$$[\mu M](dy) = \int \mu(dx) M(x, dy) \quad (\implies [\mu M](f) = \mu[M(f)])$$

- **Bayes-Boltzmann-Gibbs transformation** : $G : E \rightarrow [0, \infty[$ with $\mu(G) > 0$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)} G(x) \mu(dx)$$

E finite \iff Matrix notations $\mu = [\mu(1), \dots, \mu(d)]$ and $f = [f(1), \dots, f(d)]'$

$$\eta_n^N(f) := \frac{1}{N} \sum_{i=1}^N f(\xi_n^i) \xrightarrow{N \uparrow \infty} \eta_n(f) := \frac{\gamma_n(f)}{\gamma_n(1)}$$

with the un-normalized measures :

$$\gamma_n(f) := \mathbb{E} \left(f_n(X_n) \prod_{0 \leq p < n} G_p(X_p) \right)$$

[Potential functions G_n] & [X_n Markov chain \sim transitions M_n]

Feynman-Kac models \supset ALL of the above heuristics

- $G_n = 1_{A_n}$, $e^{-(\beta_n - \beta_{n-1})V}$, Metropolis ratio, level crossing detect.,...
- \oplus (LDP) Twisted meas. : $\mathbb{P}(V_n(X_n) \geq a)$ & $\text{Loi}((X_0, \dots, X_n) | V_n(X_n) \geq a)$

$$e^{\lambda V_n(X_n)} \mathbb{P}(X_n \in \cdot) \rightsquigarrow G_n(X_{n-1}, X_n) = e^{\lambda[V_n(X_n) - V_{n-1}(X_{n-1})]}$$

$$X_n = (X'_0, \dots, X'_n) \rightsquigarrow V_n(X_n) = \sup_{0 \leq p \leq n} U_p(X'_p)$$

A first "detailed" example

Boltzmann-Gibbs Measures :

$$\eta_n(dx) = \frac{1}{Z_n} e^{-\beta_n V(x)} \lambda(dx)$$

Feynman-Kac representation :

$$\eta_n(f) := \frac{\gamma_n(f)}{\gamma_n(1)} \quad \text{with} \quad \gamma_n(f) := \mathbb{E} \left(f_n(X_n) \prod_{0 \leq p < n} e^{-(\beta_{p+1} - \beta_p)V(X_p)} \right)$$

and

$$\mathbb{P}(X_n \in dx_n \mid X_{n-1} = x_{n-1}) = M_n(x_{n-1}, dx_n) \quad \text{with} \quad \eta_n = \eta_{n-1} M_n$$

Note :

$$\begin{aligned} Z_n &= \lambda(e^{-\beta_n V}) \\ &= \underbrace{\frac{\lambda(e^{-(\beta_n - \beta_{n-1})V} e^{-\beta_{n-1}V})}{\lambda(e^{-\beta_{n-1}V})}}_{\eta_{n-1}(e^{-(\beta_n - \beta_{n-1})V})} \times Z_{n-1} \stackrel{(\beta_0=0)}{=} \prod_{0 \leq p < n} \eta_p(e^{-(\beta_{p+1} - \beta_p)V}) \end{aligned}$$

A second "detailed" example

Restriction of measures : $A_n \downarrow$ (Ex.: $A_n = [a_n, \infty[\rightsquigarrow$ tails probab.)

$$\eta_n(dx) = \frac{1}{Z_n} 1_{A_n}(x) \lambda(dx)$$

Feynman-Kac representation :

$$\eta_n(f) := \frac{\gamma_n(f)}{\gamma_n(1)} \quad \text{with} \quad \gamma_n(f) := \mathbb{E} \left(f_n(X_n) \prod_{0 \leq p < n} 1_{A_{p+1}}(X_p) \right)$$

and

$$\mathbb{P}(X_n \in dx_n \mid X_{n-1} = x_{n-1}) = M_n(x_{n-1}, dx_n) \quad \text{with} \quad \eta_n = \eta_n M_n$$

Note :

$$\begin{aligned} Z_n &= \lambda(A_n) \\ &= \underbrace{\frac{\lambda(1_{A_n} 1_{A_{n-1}})}{\lambda(1_{A_{n-1}})}}_{\eta_{n-1}(1_{A_n})} \times Z_{n-1} \stackrel{(A_0=E)}{=} \prod_{0 \leq p < n} \eta_p(1_{A_{p+1}}) \end{aligned}$$

Path space measures = Same math. models

Historical process : $X_n := (X'_0, \dots, X'_n) \in E_n = (E'_0 \times \dots \times E'_n)$

\Downarrow

Path space particles : $\xi_n^i := (\xi_{0,n}^i, \xi_{1,n}^i, \dots, \xi_{n,n}^i) \in E_n = (E'_0 \times \dots \times E'_n)$

\Downarrow

$$\eta_n^N(f) := \frac{1}{N} \sum_{i=1}^N f_n(\xi_n^i) \longrightarrow_{N \uparrow \infty} \eta_n(f) := \frac{\gamma_n(f_n)}{\gamma_n(\mathbf{1})}$$

with the un-normalized Feynman-Kac meas. on **paths spaces** :

$$\gamma_n(f_n) = \mathbb{E} \left(f_n(X'_0, \dots, X'_n) \prod_{0 \leq p < n} G_p(X'_0, \dots, X'_p) \right)$$

Example $\hookrightarrow \eta_n = \text{Law}((X'_0, \dots, X'_n) \mid \text{without intersections})$

$$X' = \text{Random walk} \in \mathbb{Z}^d \quad \& \quad G_n(X'_0, \dots, X'_n) = 1_{\{X'_0, \dots, X'_{n-1}\}}(X'_n)$$

Flows of Feynman-Kac measures

$$\eta_n \xrightarrow{\text{Correction/mise à jour}} \hat{\eta}_n = \Psi_{G_n}(\eta_n) \xrightarrow{\text{Prédiction/exploration}} \eta_{n+1} = \hat{\eta}_n M_{n+1}$$

Nonlinear transport formulae

$$\Psi_{G_n}(\eta_n) = \eta_n S_{n,\eta_n}$$

with

$$S_{n,\eta_n}(x, \cdot) := \epsilon_n G_n(x) \delta_x + (1 - \epsilon_n G_n(x)) \Psi_{G_n}(\eta_n)$$

↓

$$\eta_{n+1} = \eta_n (S_{n,\eta_n} M_{n+1}) := \eta_n K_{n+1,\eta_n}$$

Nonlinear Markov chains $\eta_n = \text{Law}(\bar{X}_n)$ = Perfect sampling algorithm

- **Nonlinear transport formulae :**

$$\eta_{n+1} = \eta_n K_{n+1, \eta_n}$$

with the collection of Markov probability transitions :

$$K_{n+1, \eta_n} = S_{n, \eta_n} M_{n+1}$$

- **Local transitions :**

$$\mathbb{P}(\bar{X}_n \in dx_n \mid \bar{X}_{n-1}) = K_{n, \eta_{n-1}}(\bar{X}_{n-1}, dx_n) \quad \text{avec} \quad \eta_{n-1} = \text{Law}(\bar{X}_{n-1})$$

- **McKean measures (canonical process) :**

$$\mathbb{P}_n(d(x_0, \dots, x_n)) = \eta_0(dx_0) K_{1, \eta_0}(x_0, dx_1) \dots K_{n, \eta_{n-1}}(x_{n-1}, dx_n)$$

Sampling pb \Rightarrow Mean field particle interpretations

- Markov Chain $\xi_n = (\xi_n^1, \dots, \xi_n^N) \in E_n^N$ s.t.

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq_{N \uparrow \infty} \eta_n$$

- Approximated local transitions ($\forall 1 \leq i \leq N$)

$$\xi_{n-1}^i \rightsquigarrow \xi_n^i \sim K_{n, \eta_{n-1}^N}(\xi_{n-1}^i, dx_n)$$

Schematic picture : $\xi_n \in E_n^N \rightsquigarrow \xi_{n+1} \in E_{n+1}^N$

$$\begin{array}{ccc}
 \xi_n^1 & \xrightarrow{K_{n+1, \eta_n^N}} & \xi_{n+1}^1 \\
 \vdots & & \vdots \\
 \xi_n^i & \longrightarrow & \xi_{n+1}^i \\
 \vdots & & \vdots \\
 \xi_n^N & \longrightarrow & \xi_{n+1}^N
 \end{array}$$

Rationale :

$$\eta_n^N \simeq_{N \uparrow \infty} \eta_n \implies K_{n+1, \eta_n^N} \simeq_{N \uparrow \infty} K_{n+1, \eta_n} \implies \xi^i \sim \text{i.i.d. copies of } \bar{X}$$

\Downarrow

Particle McKean measures :

$$\frac{1}{N} \sum_{i=1}^N \delta_{(\xi_0^i, \dots, \xi_n^i)} \longrightarrow_{N \uparrow \infty} \text{Law}(\bar{X}_0, \dots, \bar{X}_n)$$

Feynman-Kac models \Leftrightarrow Genetic type stochastic algo.

$$\begin{bmatrix} \xi_n^1 \\ \vdots \\ \xi_n^i \\ \vdots \\ \xi_n^N \end{bmatrix} \xrightarrow{S_{n,\eta_n^N}} \begin{bmatrix} \widehat{\xi}_n^1 & \xrightarrow{M_{n+1}} & \xi_{n+1}^1 \\ \vdots & & \vdots \\ \widehat{\xi}_n^i & \xrightarrow{\quad} & \xi_{n+1}^i \\ \vdots & & \vdots \\ \widehat{\xi}_n^N & \xrightarrow{\quad} & \xi_{n+1}^N \end{bmatrix}$$

Acceptance/Rejection-Selection : [Geometric type clocks]

$$S_{n,\eta_n^N}(\xi_n^i, dx)$$

$$:= \epsilon_n G_n(\xi_n^i) \delta_{\xi_n^i}(dx) + (1 - \epsilon_n G_n(\xi_n^i)) \sum_{j=1}^N \frac{G_n(\xi_n^j)}{\sum_{k=1}^N G_n(\xi_n^k)} \delta_{\xi_n^j}(dx)$$

Ex. : $G_n = 1_A \rightsquigarrow G_n(\xi_n^i) = 1_A(\xi_n^i)$

Some key advantages

- Mean field models = **stochastic linearization/perturbation technique** :

$$\eta_n^N = \eta_{n-1}^N K_{n, \eta_{n-1}^N} + \frac{1}{\sqrt{N}} W_n^N$$

avec $W_n^N \simeq W_n$ Centered Gaussian Fields \perp .

- $\eta_n = \eta_{n-1} K_{n, \eta_{n-1}}$ stable \Rightarrow No propagation of local sampling errors
 \Rightarrow **Uniform control w.r.t. the time horizon**
- "No burning, no need to study the stability of MCMC models".
- Stochastic adaptive grid approximation
- Nonlinear system \rightsquigarrow "positive-benefic interactions.
- Simple and natural sampling algorithm.

"Asymptotic" theory: TCL, PGD, PDM, ... (n, N). some examples :

- Empirical processes :

$$\sup_{n \geq 0} \sup_{N \geq 1} \sqrt{N} \mathbb{E}(\|\eta_n^N - \eta_n\|_{\mathcal{F}_n}^p) < \infty$$

- Concentration inequalities uniform w.r.t. time :

$$\sup_{n \geq 0} \mathbb{P}(|\eta_n^N(f_n) - \eta_n(f_n)| > \epsilon) \leq c \exp -(N\epsilon^2)/(2\sigma^2)$$

+ Guionnet $\sup_{n \geq 0}$ (IHP 01) & Ledoux $\sup_{\mathcal{F}_n}$ (JTP 00) & Rio hal-09

- Propagations of chaos (+ Patras, Rubenthaler (AAP 09-10) :

$$\begin{aligned} \mathbb{P}_{n,q}^N &:= \text{Loi}(\xi_n^1, \dots, \xi_n^q) \\ &\simeq \eta_n^{\otimes q} + \frac{1}{N} \partial^1 \mathbb{P}_{n,q} + \dots + \frac{1}{N^k} \partial^k \mathbb{P}_{n,q} + \frac{1}{N^{k+1}} \partial^{k+1} \mathbb{P}_{n,q}^N \end{aligned}$$

with $\sup_{N \geq 1} \|\partial^{k+1} \mathbb{P}_{n,q}^N\|_{\text{tv}} < \infty$ & $\sup_{n \geq 0} \|\partial^1 \mathbb{P}_{n,q}\|_{\text{tv}} \leq c q^2$.

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- 3 Additive functionals**
- 4 Normalizing constant estimation
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- Path space models $\mathbb{P}_n := \text{Law}(X_0, \dots, X_n)$

$$d\mathbb{Q}_n := \frac{1}{Z_n} \left\{ \prod_{0 \leq p < n} G_p(X_p) \right\} d\mathbb{P}_n$$

- Hyp. : $M_n(x_{n-1}, dx_n) = H_n(x_{n-1}, x_n) \lambda_n(dx_n)$

$$\Rightarrow \mathbb{Q}_n(d(x_0, \dots, x_n)) = \eta_n(dx_n) M_{n, \eta_{n-1}}(x_n, dx_{n-1}) \dots M_{1, \eta_0}(x_1, dx_0)$$

with the backward transitions :

$$M_{p+1, \eta}(x, dx') \propto G_p(x') H_{p+1}(x', x) \eta(dx')$$

- Particle estimates \sim complete genealogical tree :

$$\mathbb{Q}_n^N(d(x_0, \dots, x_n)) = \eta_n^N(dx_n) M_{n, \eta_{n-1}^N}(x_n, dx_{n-1}) \dots M_{1, \eta_0^N}(x_1, dx_0)$$

2 type of path-space estimates

- **Complete genealogical tree** \implies McKean meas. \oplus FK-Path space

$$\frac{1}{N} \sum_{i=1}^N \delta_{(\xi_0^i, \dots, \xi_n^i)} \simeq_N \text{Loi}(\bar{X}_0, \dots, \bar{X}_n) \quad \& \quad \mathbb{Q}_n^N \simeq_N \mathbb{Q}_n$$

- **Simple genealogical tree** \implies FK-Path space

$$\eta_n^N = \frac{1}{N} \sum_{i=1}^N \delta_{(\xi_{0,n}^i, \xi_{1,n}^i, \dots, \xi_{n,n}^i)} \simeq_N \mathbb{Q}_n = \eta_n$$

Some estimates

Additive functional (with Doucet & Singh Hal-INRIA july 09) :

$$F_n(x_0, \dots, x_n) = \frac{1}{n+1} \sum_{0 \leq p \leq n} f_p(x_p)$$

- Bias estimate + uniform \mathbb{L}_p -bounds + variance

$$N \mathbb{E} \left([(\mathbb{Q}_n^N - \mathbb{Q}_n)(F_n)]^2 \right) \leq c \times (1/n + 1/N)$$

- Uniform exponential concentration

$$\frac{1}{N} \log \sup_{n \geq 0} \mathbb{P} \left(|[\mathbb{Q}_n^N - \mathbb{Q}_n](F_n)| \geq \frac{b}{\sqrt{N}} + \epsilon \right) \leq -\epsilon^2 / (2b^2)$$

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 - A key multiplicative formula
 - Some examples
 - Convergence analysis
- 5 Some references

Problem : Un-normalized measures estimation

$$\gamma_n(f) := \mathbb{E} \left(f_n(X_n) \prod_{0 \leq p < n} G_p(X_p) \right) \simeq_{N \uparrow \infty} \gamma_n^N(f) := ???$$

Key observation :

$$\eta_n(G_n) \gamma_n(1) = \gamma_n(G_n) = \gamma_{n+1}(1)$$

⇒ Multiplicative formula \rightsquigarrow **unbias** particle estimates

$$\begin{aligned} \gamma_n(1) &= \prod_{0 \leq p < n} \eta_p(G_p) \longleftarrow_{N \uparrow \infty} \gamma_n^N(1) := \prod_{0 \leq p < n} \eta_p^N(G_p) \\ &\downarrow \\ \gamma_n(f) &:= \gamma_n(1) \times \eta_n(f) \longleftarrow_{N \uparrow \infty} \gamma_n^N(f) := \gamma_n^N(1) \times \eta_n^N(f) \end{aligned}$$

Note. : If G_n takes null values (ex. $G_n = 1_A$) \Rightarrow convention = we estimate by 0.

2 Examples : Feynman-Kac models

- **Tube confinement :**

$$\gamma_n(1) \stackrel{G_n=1_A}{=} \mathbb{P}(\cap_{0 \leq p < n} X_p \in A) \simeq_{N \uparrow \infty} \gamma_n^N(1) := \prod_{0 \leq p < n} \eta_p^N(A)$$

- **Self avoiding walks :**

$$\gamma_{n+1}(1) = \mathbb{P}(\forall p < q \leq n \ X_p \neq X_q) = \frac{1}{(2d)^n} \text{Card} \{ \text{s.a.w. with length } = n \}$$
$$\simeq_{N \uparrow \infty}$$

$$\gamma_{n+1}^N(1) := \prod_{0 \leq p \leq n} \text{empirical mean potential at time } p$$

Several strategies :

- 1 Path evolutions with G -intersection detection
- 2 Local transitions without intersections with \widehat{G} -future proba. of intersection.
- 3 ...

+2 Examples : Boltzmann-Gibbs static measures

- **Partition functions:** $(G_n = e^{-(\beta_{n+1}-\beta_n)V})$ et $(\eta_n M_n = \eta_n) \Rightarrow d\eta_n \propto e^{-\beta_n V} d\lambda$

$$(\rightsquigarrow \text{ Note : } \lambda(e^{-\beta_n V}) = \lambda(G_n \times e^{-\beta_{n-1} V}) = \eta_n(G_n) \lambda(e^{-\beta_{n-1} V}))$$

\Downarrow

$$\lambda(e^{-\beta_n V}) = \gamma_n(\mathbf{1}) \simeq_{N \uparrow \infty} \gamma_n^N(\mathbf{1}) := \prod_{0 \leq p < n} \eta_p^N(e^{-(\beta_{p+1}-\beta_p)V})$$

- **Volumes and Cardinals :**

$$(G_n = 1_{A_{n+1}}) \quad \text{and} \quad (\eta_n M_n = \eta_n) \implies \eta_n(dx) \propto 1_{A_{n+1}} \lambda(dx)$$

\Downarrow

$$\lambda(A_n) = \gamma_n(\mathbf{1}) \simeq_{N \uparrow \infty} \gamma_n^N(\mathbf{1}) := \prod_{0 \leq p \leq n} \eta_p^N(A_{p+1})$$

Convergence analysis

- Asymptotic theory : fluctuations & deviations
+ A. Guionnet (AAP 99, SPA 98), + L. Miclo (SP 2000), + D. Dawson
- Non asymptotic theory : bias and variance estimates

- Taylor type expansion (+Patras & Rubenthaler (AAP 09)) :

$$\mathbb{E} \left((\gamma_n^N)^{\otimes q}(F) \right) =: \mathbb{Q}_{n,q}^N(F) = \gamma_n^{\otimes q}(F) + \sum_{1 \leq k \leq (q-1)(n+1)} \frac{1}{N^k} \partial^k \mathbb{Q}_{n,q}(F)$$

[Hyp. \sim simple genetic algo. $\epsilon_n = 0$ & potentiel > 0]

- Variance estimates (+Cerou & Guyader Hal-INRIA nov.08) :

$$\mathbb{E} \left([\gamma_n^N(f_n) - \gamma_n(f_n)]^2 \right) \leq c \frac{n}{N} \times \gamma_n(1)^2$$

Hyp. + weak conditions \supset :

- Mean field models with acceptance rate $\forall \epsilon_n \geq 0$.
- Potential functions ≥ 0 (\supset indicator functions).
- Path space models $X_n = (X'_0, \dots, X'_n)$, with X' "mixing".

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- 4 Normalizing constant estimation
- 5 Some references**

Particle methods & Sequential Monte Carlo

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